Exam. Code : 211001
Subject Code : 5474

M.Sc. (Mathematics) I^{st} Semester

ALGEBRA—I

Paper—MATH-553

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt **TWO** questions from each Unit. Each question carries equal marks.

UNIT—I

1. Prove that a finite semi-group G is a group if and only if G satisfies both cancellation laws.
2. State and prove Lagrange's Theorem.
3. If p is the smallest prime factor of the order of a finite group G, prove that any subgroup of index p is normal in G.
4. (a) Find all the subgroups of $\mathbb{Z}/21\,\mathbb{Z}$.
   (b) If H is a subgroup of a group G such that $x^2 \in H\ \forall\ x \in G$ then show that H is normal subgroup of G.

UNIT—II

5. (a) Show that for $G = S_3$ then G', commutator subgroup of G, is A_3.
   (b) Let G be a group of order 231, show that Sylow 11-subgroup of G is contained in Z(G), centre of G.

6. State and prove Fundamental Theorem of Homomorphism for groups.

7. (a) Let G be a group such that G/Z(G) is cyclic show that G is abelian.

   (b) Show that a cyclic group of order 8 is homomorphic to a cyclic group of order 4.

8. (a) For any group G, prove that In(G) ≅ G/Z(G).

   (b) Prove that the group of automorphisms of a cyclic group is abelian.

UNIT—III

9. (a) Prove that any two disjoint permutations commute.

   (b) Show that A_4 is the only subgroup of order 12 in S_4.

10. Prove that the set A_n of all even permutations of degree n forms a finite group of order $\frac{\lfloor n}{2}$ with respect to permutation multiplication.

11. (a) Show that the group $\langle \mathbb{Z}/\langle 8 \rangle, + \rangle$ cannot be written as the direct sum of two non-trivial subgroups.

    (b) Show that the external direct product of additive group of integers $\mathbb{Z}$ with itself is not a cyclic group.

12. Prove that the number of non-isomorphic abelian groups of order p^n (p prime) is equal to the number of partition of n.

UNIT—IV

13. State and prove Sylow's First Theorem.

14. Show that a group of order $p^2.q$ is solvable, where p and q are prime numbers.

15. (a) Write down all composition series for Q_8.

    (b) Show that every group of order 15 is cyclic.

16. Verify class equation for S_3.

UNIT—V

17. (a) In a ring R, $x^3 = x$ for all $x \in R$, then show that R is a commutative ring.

    (b) Give an example of an ideal I of a ring R such that I is left ideal but not right ideal.

18. (a) Prove that every finite integral domain is a field.

    (b) Prove or disprove there is an integral domain with six elements.

19. (a) Show that $M_2(\mathbb{R})$, the ring of all 2×2 matrices over the field of real numbers is simple.

    (b) Find all homomorphism from ring $\mathbb{Z}$ onto $\mathbb{Z}$.

20. (a) If every ideal of a commutative ring R with unity is prime, show that R is a field.

    (b) Show that ring $2\mathbb{Z}$ is not isomorphic to ring $5\mathbb{Z}$.